XI. Eigensystems

An eigensystem is defined by the equation

$$Ax = \lambda x$$  \hspace{1cm} (1)

where $A$ is a square matrix, $x$ is a vector, and $\lambda$ is a scalar. In other words, the transformation $Ax$ results in a simple scaling of $x$. Given a normal matrix (a class of matrix that includes symmetric and orthogonal matrices), $A$ we can always find a set of $\lambda$’s (eigenvalues) and a corresponding set of $x$’s (eigenvectors).

The eigenvectors are equivalent to modes of physical systems. Consider the transverse oscillations of beads on a string (Figure 1). The two beads have mass $m$, and are separated by flexible strings of length $l$ when at equilibrium. Suppose displacements $x_n$ of the beads are so small that the tension $T$ in the strings can be taken to be constant. The angle of each string to the horizontal is $\theta_n$ as illustrated in the figure. The equation of motion for the displacement $x_1$ of the first bead is

$$m \frac{d^2x_1}{dt^2} = -T \sin \theta_1 + T \sin \theta_2$$  \hspace{1cm} (2)

Under the assumption that the displacements are small, $\sin \theta_n$ may be approximated as $\tan \theta_n$ so

$$m \frac{d^2x_1}{dt^2} = -T \frac{x_1}{l} + T \frac{x_2 - x_1}{l}$$  \hspace{1cm} (3)

Rearranging produces

$$m \frac{d^2x_1}{dt^2} = \frac{T}{l} (x_2 - 2x_1)$$  \hspace{1cm} (4)

Similarly, the equation of motion for the second bead is

$$m \frac{d^2x_2}{dt^2} = \frac{T}{l} (x_1 - 2x_2)$$  \hspace{1cm} (5)

Proceed by assuming a solution of the form

$$x_n = X_n e^{i\omega t}$$  \hspace{1cm} (6)

Figure 1. Beads on a string at equilibrium (top) and displaced (bottom).
Then the pair of equations to be solved are

\[
\begin{align*}
(2 - \lambda)X_1 - X_2 &= 0 \\
-X_1 + (2 - \lambda)X_2 &= 0
\end{align*}
\]  

(7)

where \( \lambda = \frac{\omega^2 ml}{T} \) are eigenvalues. For a nontrivial solution to this problem, we must have

\[
\det \begin{bmatrix} 2 - \lambda & -1 \\ -1 & 2 - \lambda \end{bmatrix} = 0
\]

(8)

So

\[(2 - \lambda)^2 - 1 = 0\]

(9)

which has the two solutions \( \lambda = 1, 3 \). For \( \lambda = 1 \)

\[
X_1 - X_2 = 0
\]

(10)

which says simply that \( X_1 = X_2 \). Writing this solution as a normalized vector

\[
x = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\]

(11)

which is the eigenvector for the eigenvalue \( \lambda = 1 \). For \( \lambda = 3 \)

\[
-X_1 - X_2 = 0
\]

(12)

with the normalized solution

\[
x = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}
\]

(13)

The two solutions, commonly referred to as modes, as shown in Figure 2.

Given an initial specification of bead positions in terms of modes we can predict the evolution of the system. While equations of motion of the beads (4-5) are coupled, the equations

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**Figure 2.** Modes of a two-beaded string. Mode 1 (top) has both beads moving in phase, and mode 2 (bottom) has the beads out of phase.
for the modes are uncoupled. That is, the modes evolve independently of each other, and the evolution of the system is a linear combination of the two modes.

Note that the equation we solve was in the form (1) with

\[
A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}
\]  \hspace{1cm} (14)

So what we were doing in solving the problem of beads on a string, was precisely the solution of an eigensystem. In linear vector space language, the modes are the most convenient coordinates. The eigensystem can be written

\[
AP = PD
\]  \hspace{1cm} (15)

where \( D \) is diagonal with the eigenvalues along the diagonal, and the columns of the orthogonal matrix \( P \) are the eigenvectors. A few standard relations are

\[
D = P^T AP\hspace{1cm} (16)
\]

\[
A = PDP^T\hspace{1cm} (17)
\]

\[
A^{-1} = PD^{-1}P^T\hspace{1cm} (18)
\]

Given the eigenvalue decomposition of a matrix, (18) gives an easy way of determining invertibility simply by determining whether any of the eigenvalues are zero. The ratio of the smallest to largest eigenvalue, referred to as the condition number, is an indication of the stability of the inversion to numerical error.